P, PI, and PID Control

P stands for Proportional

You have learned that a proportional controller leaves a steady state error or offset. To eliminate this error, integral action is added to the controller. To understand how integral control works, we need to study the integrator implemented using a simple op-amp integrator.

Integrator Circuit

For a capacitor

$$v \propto q = \frac{1}{C} q$$

$$\frac{dv}{dt} = \frac{1}{C} \frac{dq}{dt}$$

$$\frac{dq}{dt} = i \text{ , } v = \frac{q}{C}$$

$$\frac{dv}{dt} = \frac{1}{c} i_C$$

Therefore,

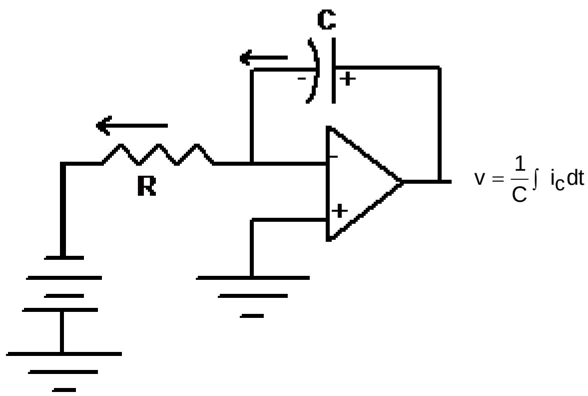
$$i_C = C \frac{dv}{dt}$$

or

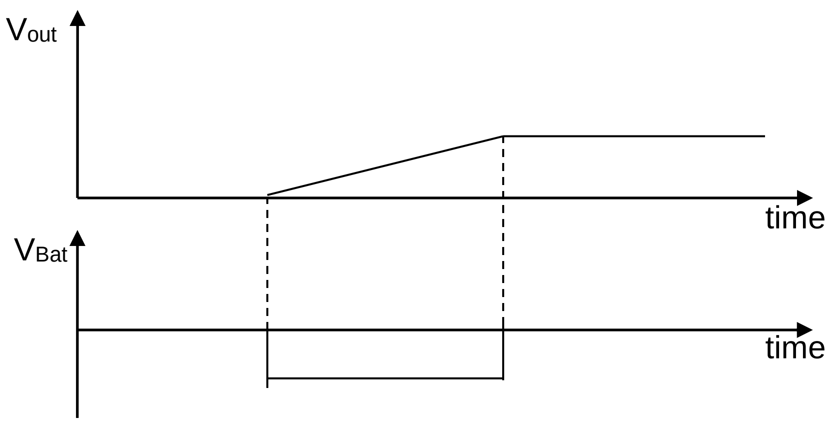
$$dv = \frac{1}{C} i_C dt$$

or

$$v = \frac{1}{C} \int i_C dt$$



For a constant voltage, the current into the resistor is constant and the current charging the capacitor is constant. Then the integration acting on the constant current produces a ramping voltage



Notice that when the battery voltage is at 0 Volts, the integrating action stops and the output voltage remains constant. That is the capacitor stops charging. The output will change as long as the battery voltage is not zero. Of course if the output reaches the power supply voltage , the output will saturate.

In a PI controller, the battery voltage shown in the op-amp circuit is really a voltage proportional to error and as long as an error exists, the integrator changes is output causing the controlled variable to change to reduce the error. *Integration only stops when the error is eliminated*.

La Place Transform of the Integrator

$$V = \frac{1}{C}\int i\,dt$$

or

for the Op Amp integrator

$$V_o = \frac{1}{C}\int \frac{V}{R}dt$$

or

$$V_o = \frac{1}{RC}\int V\,dt$$

Remember the following LaPlace Transforms

$$\frac{d^2e(t)}{dt^2} \Leftrightarrow s^2E(s) \;\text{ or }\; s^2E(s)$$

$$\frac{de(t)}{dt^2} \Leftrightarrow sE(s) \;\text{ or }\; s^1E(s)$$

$$e(t) \Leftrightarrow E(s) \;\text{ or }\; s^0E(s)$$

$$\int e(t) \Leftrightarrow \frac{E(s)}{s} \;\text{ or }\; s^{-1}E(s)$$

Then Laplace transform for the Op Amp Integrator

$$V_o = \frac{1}{RC}\int V\,dt \Leftrightarrow \frac{1}{RC}\cdot\frac{V}{s}$$

if $\tau_i = RC$

$$V_o = \frac{1}{\tau_i}\int V\,dt \Leftrightarrow \frac{V}{\tau_i s}$$

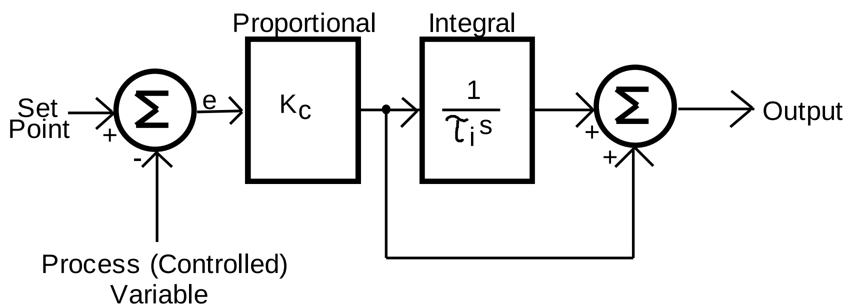
The Proportional plus Integral (PI) Controller

The time domain expression for the PI controller is:

$$\text{Cont Out} = K_C e + \frac{K_C}{\tau_i}\int e\,dt$$

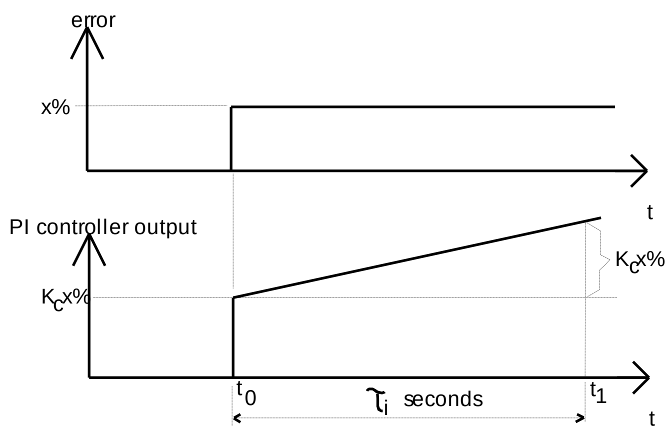
The block diagram using LaPlace transforms is shown below.

Block Diagram of PI Controller



A term often used to describe the magnitude of integral action is time/repeat, often expressed is **minutes per repeat**. This number is nothing more than the integrator time. The term is an old way of expressing integrator action and it is sometimes describes as the amount of time that integration would take to repeat the action of proportional if the error seen by the controller was constant (this could only configured in a lab setting).

Minutes/Repeat
(Integrator Description Term)



$$\text{Cont Out} = K_C e + \frac{K_C}{\tau_i}\int e\,dt$$

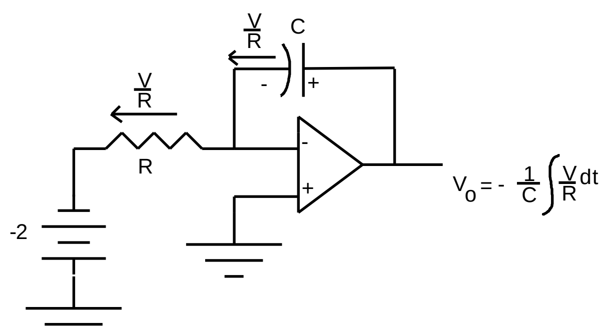
if a constant error of x% was sent to the input of the PI controller, then the integrator output could be expressed as

$$\frac{K_C}{\tau_i}\int e\,dt \quad \text{or} \quad \frac{K_C}{\tau_i}\int x\,dt \quad \text{or} \quad \frac{K_C}{\tau_i}x(t_1 - t_0)$$

$$\text{if } (t_1 - t_0) = \tau_i, \text{ then } \frac{K_C}{\tau_i}x(t_1 - t_0) = \frac{K_C}{\tau_i}x \cdot \tau_i = K_C \cdot x$$

That is, for a constant error of x%, the integral action will repeat the proportion al action in a time of τ_i sec s.We say the integral action is τ_i sec s per repeat. Of course the expression could be in minutes per repeat.

Example: In the analog PI controller, the set point = the controlled variable. Suppose the PV drops by 1 volt (the gain of the controller is 2), how long will it take for the integrator to change the output of the controller by the same amount as proportional action alone.



$$V_o = -\frac{1}{RC}\int_{t_0}^{t_1} V\,dt = -\frac{1}{RC}\int_{t_0}^{t_1}(-2)\,dt$$

$$\text{if } \tau_i = RC$$

$$V_o = \frac{1}{\tau_i}(-2)t\Big]_{t_0}^{t_1} = \frac{1}{\tau_i}(-2)(t_0 - t_1)$$

$$\text{if } t_0 - t_1 = \tau_i \text{ then } V_o = \frac{1}{\tau_i}(-2)\cdot\tau_i = -2 \text{ volts}$$

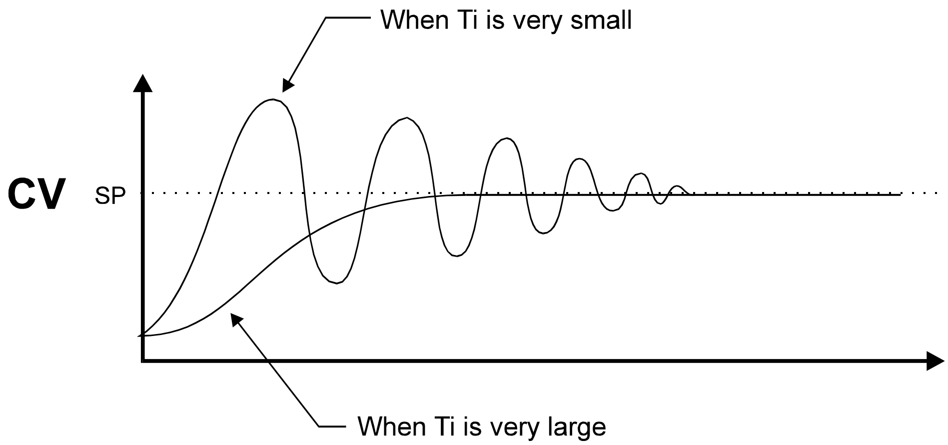
- In a time equal to Ti (integrator time), integral action has changes the controller output by the same amount as the proportional

- We say that integral action is specified in units of **Time/Repeat.**

PI Controller Summary

- **Proportional** only control leaves a **steady state error** or **offset**.
- **Integral** action integrates the error and continues changing the controller output as long as an error exists. Eventually the **error** will be **eliminated**.
- **Integral** action tends to **destabilize** a process (make the process more oscillatory).

Response of PI Controller System with 2 Very Different Integrator Settings



A **large integrator time T_i** or (τ_i) results in **a small amount of integrator action**.

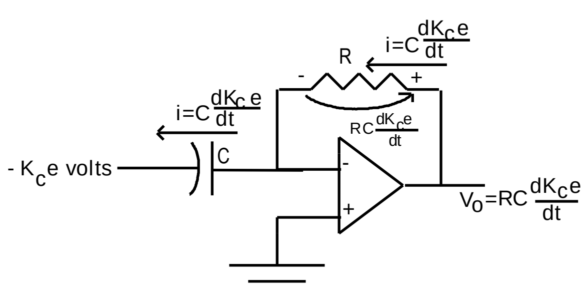
To reduce oscillations and overshoot produced by use of high controller gain(low proportional band) or small integrator time or both, derivative is often added.

Derivative

Derivative action stabilizes a controller or reduces the tendency towards oscillation.

Adding derivative results in a controller configuration called the **PID** or **Proportional Integral Derivative** Controller. This is sometimes referred to as a **3 mode controller** and is the most commonly used configuration in industrial control

To explain derivative action, we will use an op amp circuit and R,C components.



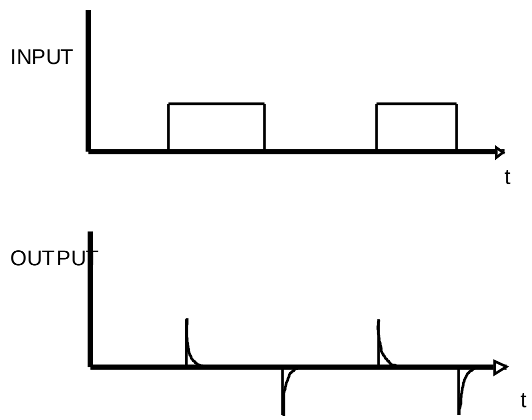
This is called a differentiator circuit.

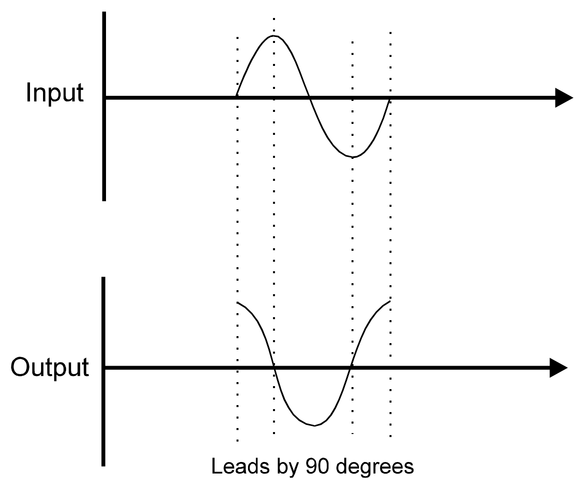
The current through the capacitor of the differentiator is

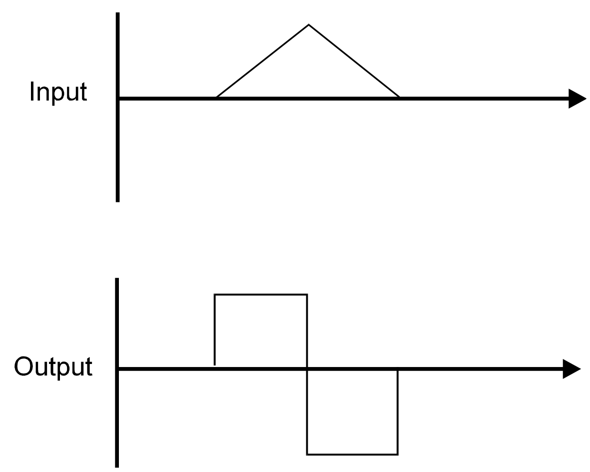
$$i = C\frac{dK_C e}{dt} \Leftrightarrow i = CK_C\frac{de}{dt}$$

Current will only flow when there is a change in voltage across the capacitor. The waveforms below illustrate the derivative action produced by the op-amp differentiator

Derivative Action:



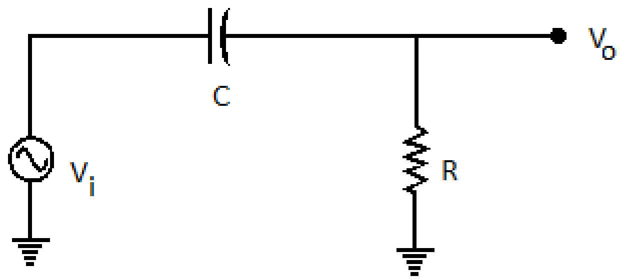




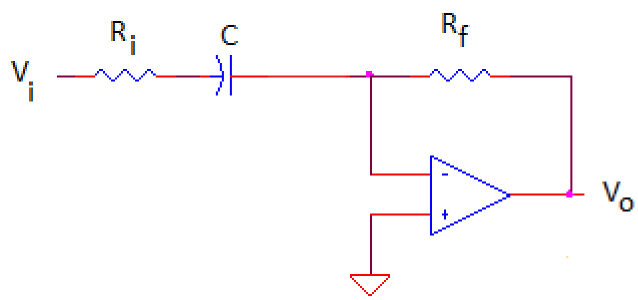
Derivative Action:

- Supresses overshoot
- Similar to action of a shock absorber
- Shock absorber provides mechanical derivative action
- Differentiation is sensitive to niose
- Differentiator acts like a high pass filter (see diagram below)

Remember, reactance of capacitor is $X_C = \frac{1}{2\pi fC}$



A practical differentiator limits the high frequency gain:



At high frequencies gain is limited to $-\frac{R_f}{R_i}$

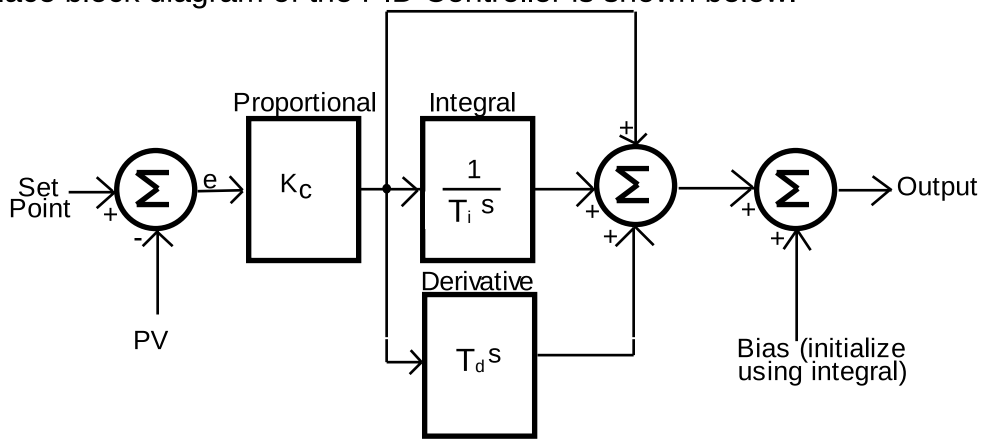
If v_i were a sine wave source, as the frequency increases, the reactance of the capacitor decreases. At very high frequencies, Ri will limit the current as Xc goes to 0

Three Mode Controller

Time domain equation of PID controller is shown below.

$$Cout = K_c e + \frac{K_c}{Ti}\int e\,dt + K_c T_d \frac{de}{dt}$$

The LaPlace block diagram of the PID Controller is shown below.



Op AMP PID Controller (error amplifier not shown)