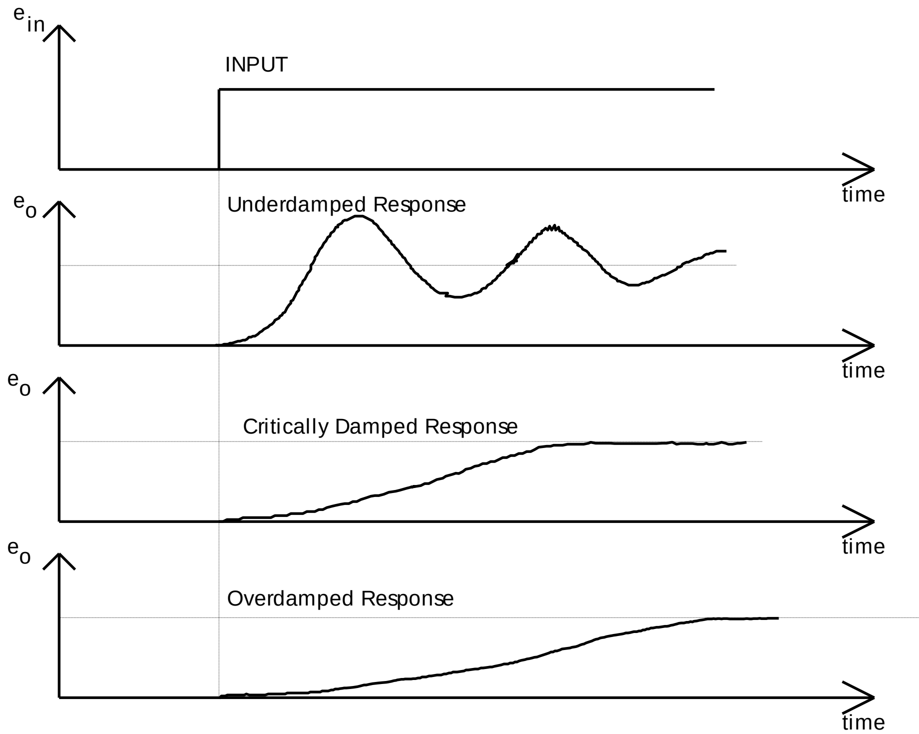
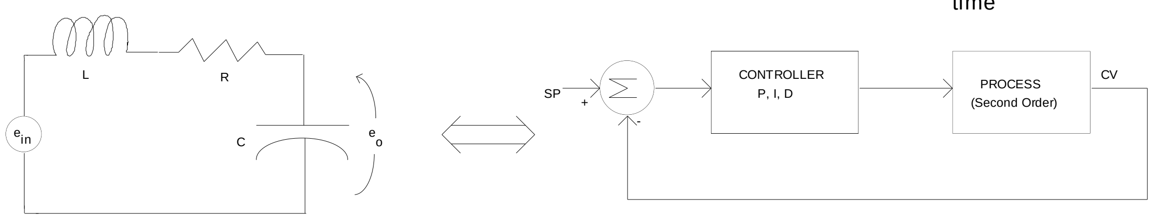
Why Study the RLC Circuit?

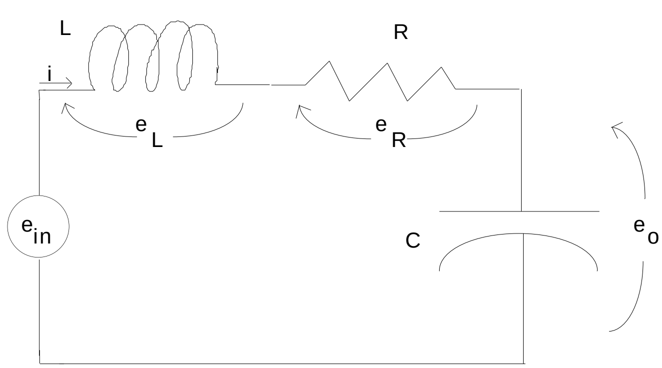
The RLC circuit and the closed loop feedback control system have similarities in the way they respond to step input changes. Shown below is the 3 types of response that an RLC circuit displays to step changes in input voltage.





For a step change in SP, feedback control system controlled variable CV can behave in a similar manner to behaviour of e_o for a step change in e_{in}.

That is e_o, and CV can exhibit underdamped, critically damped, or overdamped response.



$$i = C\frac{de_o}{dt}$$

$$e_R = Ri = RC\frac{de_o}{dt}$$

$$e_L = L\frac{di}{dt} = L\frac{d(C\frac{de_o}{dt})}{dt} = LC\frac{d^2e_o}{dt^2}$$

$$e_i = e_L + e_R + e_o$$

$$e_i = LC\frac{d^2e_o}{dt^2} + RC\frac{de_o}{dt} + e_o$$

Find the transfer function $\frac{E_o(s)}{E_i(s)}$

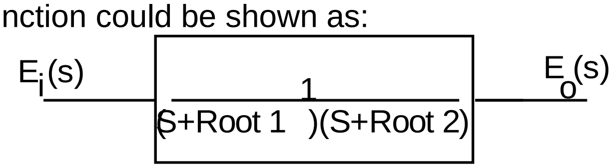
$$E_i(s) = LCS^2E_o(s) + RCE_oS(s) + E_o(s)$$

$$E_i(s) = E_o(s)[LCS^2 + RCS + 1]$$

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCS^2 + RCS + 1}$$

The denominator $LCS^2 + RCS + 1$ is sometimes referred to as the characteristic equation. It is a quadratic equation whose factors can be real, and different, real and equal, complex, or imaginary depending on values of L, R, and C.

That is the transfer function could be shown as:



Find the roots of the characteristic equation for the following conditions.
Remember the quadratic formula for $ax^2+bx+c=0$.

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Factoring the characteristic equation using the quadratic formula:

$$S = \frac{-RC + \sqrt{R^2C^2 - 4LC}}{2LC} \text{ or } S = \frac{-RC - \sqrt{R^2C^2 - 4LC}}{2LC}$$

When $R^2C^2 > 4LC$, the roots are Real and unequal. That is:

$$R > 2\sqrt{\frac{LC}{C^2}}$$

When $R^2C^2 = 4LC$, the roots are Real and equal. That is:

$$R = 2\sqrt{\frac{LC}{C^2}}$$

When $R^2C^2 < 4LC$, the roots are Complex.

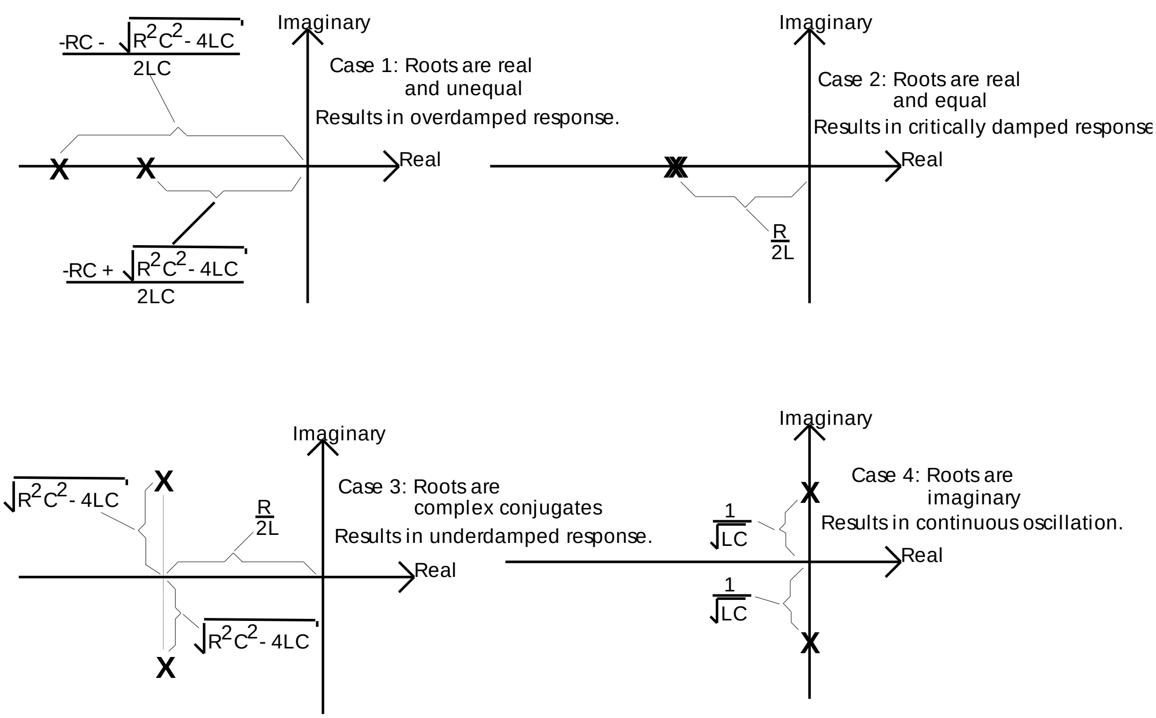
$$R < 2\sqrt{\frac{LC}{C^2}}$$

That is, the term under the square root is Imaginary. We say that the roots are complex conjugates.

When R=0, we get purely imaginary roots. That is, the term under the square root is:

$$\sqrt{-\frac{1}{LC}} \text{ or } j\sqrt{\frac{1}{LC}} \text{ where j is } \sqrt{-1}$$

We can summarize the characteristics of the roots by plotting the real and imaginary parts of the roots on a graph whose y axis is the imaginary part of the root, and whose x value is the real part of the root. We show the root with an X and refer to it as a pole of the transfer function.



As shown on the above drawing,

1) Real and different roots (case 1) result in a **step** response that is overdamped.

2) Real and equal roots (case 2) results in a **step** response that is critically damped.

3) Complex conjugate roots (case 3) result in a **step** response that is overdamped.

4) Purely imaginary roots (case 4) result in a **step** response that is continuously oscillating.

In the same way that RLC circuit behaves as described with the 4 cases above, ***a closed loop feedback control system can exhibit underdamped, critically damped, overdamped behaviour, and continuous oscillatory behaviour.***

Of course a control system must never be allowed to oscillate continuously. In fact, severely underdamped behaviour is undesirable as well.

Root Locus Plot.

A root locus plot for the RLC circuit is a drawing that shows the roots (poles) of the characteristic equation on a Real Imaginary axis diagram (sometimes called the S plane or frequency plane) as a parameter --- in this case R ---- is varied from say a very large value to 0.

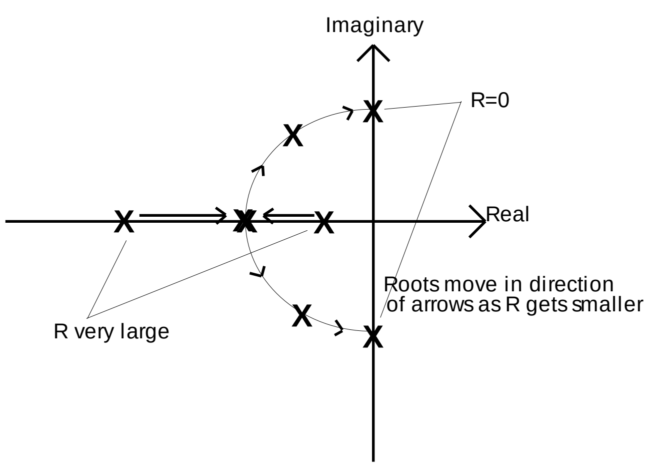
For the RLC, when the value of R is very large (say approaching infinity), the roots are real and unequal with one root (pole) being very far from the origin on the real axis and one root being very close to the origin. This response is highly overdamped and tends to be similar to the step response of a first order system such as an RC circuit.

As R gets smaller the two real roots start to move closer to each other. When the roots lie on top of each other, this is the condition of both roots being real and equal. The step response for this condition is critically damped.

If R continues to get smaller, the roots become complex conjugates and the step response exhibits underdamped behaviour.

When R equals zero the roots are both purely imaginary and the step response causes continuous oscillation.

In a similar way, when a control systems P, I, or D is changed it causes roots of the characterisitic equation of the overall process control system to change. This can lead to the response of the controlled variable to step changes in the set point that behave in a similar way to the RLC. That is the step response may be overdamped, critically damped, underdamped, or continuously oscillating.



Root Locus Plot
For RLC circuit as R is
Varied From a Large Value
To Zero

General Form of 2'nd Order Transfer Function

The general form of the second order transfer function can be used to describe the step response of a second order system. This is handy as the general form can be used for electromechanical systems, thermodynamic systems, hydraulic systems or an electronic system such as the RLC.

Remembering the transfer function for the RLC circuit:

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCS^2 + RCS + 1}$$

The general form of the 2'nd order transfer function is:

$$\frac{1}{(\frac{1}{\omega_n^2})S^2 + 2(\frac{\rho}{\omega_n})S + 1}$$

ω_n is the ***undamped*** natural frequency

ρ is the damping coefficient

If you compare the coefficients of the characteristics of the denominator of the transfer function for the RLC with the denominator of the general form:

$$\frac{1}{\omega_n^2} = LC, \qquad 2(\frac{\rho}{\omega_n}) = RC$$

$$\text{then} \quad \omega_n = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f = \frac{1}{2\pi\sqrt{LC}}$$

$$Q \quad RC = \frac{2\rho}{\frac{1}{\sqrt{LC}}}$$

$$\therefore \quad \rho = \frac{RC}{2\sqrt{LC}} \quad \text{or} \quad \rho = \frac{R}{2}\sqrt{\frac{C}{L}}$$